

COMMUTATORS AND LINEAR SPANS OF PROJECTIONS IN CERTAIN FINITE C*-ALGEBRAS

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ABSTRACT. Assume that \mathcal{A} is a unital separable simple C*-algebra with real rank zero, stable rank one, strict comparison of projections, and that its tracial simplex $T(\mathcal{A})$ has a finite number of extremal points. We prove that every self-adjoint element a in \mathcal{A} with $\tau(a) = 0$ for all $\tau \in T(\mathcal{A})$ is the sum of two commutators in \mathcal{A} and that every positive element of \mathcal{A} is a linear combination of projections with positive coefficients. Assume that \mathcal{A} is as above but σ -unital. Then an element (resp. a positive element) a of \mathcal{A} is a linear combination (resp. a linear combination with positive coefficients) of projections if and only if $\bar{\tau}(R_a) < \infty$ for every $\tau \in T(\mathcal{A})$, and if and only if , where $\bar{\tau}$ denotes the extension of τ to a tracial weight on \mathcal{A}^{**} and $R_a \in \mathcal{A}^{**}$ denotes the range projection of a . Assume that \mathcal{A} is unital and as above but $T(\mathcal{A})$ has infinitely many extremal points. Then \mathcal{A} is not the linear span of its projections. This result settles two open problems of Marcoux in [30].

1. INTRODUCTION

In the history of Operator Theory and Operator Algebras the study of how bounded operators are composed of the fundamental building blocks, projections, has attracted many researchers' attention.

In 1967 Fillmore [18] found that every bounded operator on a separable Hilbert space is made up of a linear combination of 257 projections. Soon after, the number of needed projections was reduced to 16 by Percy and Topping [33] via Brown and Percy's characterization of commutators [7] and more recently to 10 by Matsumoto [31].

Percy and Topping ([33] and [34]) proved that every element in properly infinite von Neumann algebras or in certain type II_1 factors (Wright factors) can be decomposed as a (finite) linear combination of projections. The same result was proven for the harder case of all type II_1 von Neumann algebras by Fack and De La Harpe [17] and then Goldstein and Paszkiewicz [21] proved that the same holds if and only if the von Neumann algebra does not have a finite type I direct summand with infinite dimensional center.

Of course the same conclusion cannot hold for all C*-algebras given the lack of projections in some algebras (e.g., see Blackadar [1]). Thus, the following question arises naturally:

(ALP): Which C*-algebras are the (algebraic) linear span of projections? And if an algebra is not the linear span of its projections, how to characterize elements that are linear combination of projections?

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It is natural to focus first on the algebras with the highest level of abundance of projections, namely, C^* -algebras of real rank zero ([10]). Every self-adjoint element in a C^* -algebra of real rank zero can be approximated by linear combinations of mutually orthogonal projections, namely elements with finite spectrum.

Marcoux has proved ([28] and [29], see also his survey [30]) that the following simple unital C^* -algebras are the linear span of projections.

- Simple purely infinite ones.
- AF-algebras with finitely many extremal tracial states.
- AT-algebra with real rank zero and finitely many extremal tracial states.
- Certain AH-algebras with real rank zero, bounded dimension growth, and finitely many extremal tracial states.

Related to the (ALP) question is the following non-trivial question:

(PCP): For which C^* -algebras are all *positive* elements linear combinations of projections with positive coefficients? (*positive combinations of projections* for short). And if not all, how to characterize positive elements that are positive combination of projections?

Fillmore observed ([18]) that a positive infinite rank compact operator in $B(H)$ cannot be a positive combination of projections; Fong and Murphy proved ([19]) that these are the only bounded operators which are not.

An analogous result holds in σ -finite type II_∞ von Neumann factors where we proved in [26, Corollary 3.5] that all positive elements are positive combinations of projections except those that have infinite range projection and belong to the Breuer ideal generated by all finite projections. Moreover, all positive elements in a von Neumann factor of type I_n , II_1 , or σ -finite type III are positive combinations of projections ([26, Theorem 2.12]). In the non σ -finite case or in von Neumann algebras with a nontrivial center, a necessary and sufficient condition for a positive element to be a positive combination of projections is given in terms of central ideals and the central essential spectrum ([26, Theorem 2.12]).

As usual, the purely infinite case is more tractable. We did prove in [25] that all positive elements in purely infinite simple C^* -algebras or in their multiplier algebras are positive linear combinations of projections. Note that all purely infinite simple C^* -algebras have real rank zero ([44]). Finite C^* -algebras of real rank zero, however, are considerably harder.

In this article our targets are simple separable C^* -algebras with real rank zero, stable rank one, which have strict comparison of projections and have finitely many extremal tracial states. It may be somewhat surprising that the determining factor turns out to be the number of extreme points in $T(\mathcal{A})$, which is a w^* -compact simplex. We will show that if the number of extreme points is infinite then there are positive elements that are not linear combinations of projections (Proposition 5.1); this settles two questions by Marcoux [30]. If the number of extreme points is finite, then every element in the algebra is a linear combination of projections (Theorem 4.4) and every positive element is a positive linear combination of projections (Corollary 6.5). This subsumes the above mentioned results for finite algebras by Marcoux. For non-unital algebras we will provide in Theorem 6.1 and Corollary 6.6 necessary and sufficient conditions for an element (resp. a positive element) to be a linear combination (resp. a positive combination) of projections. This shows

that neither the ALP property nor the PCP property are invariant under Morita equivalence.

Key ingredients in our proofs are

- Embedding in \mathcal{A} a unital simple AH-algebra \mathcal{C} with real rank zero and dimension growth bounded by 3 and having the same K-theory invariants, based on a result of Lin ([27]) and the Elliott-Gong classification ([14]);
- Extending the construction of Fack ([16]) and Thomsen ([36]) via certain inductive limit to approximate elements with zero traces by a bounded number of commutators;
- Marcoux's technique ([29]) to express those element as sum of commutators first, and then as linear combination of projections;
- Brown's interpolation property ([9]);
- the extension of traces on \mathcal{A} to tracial weights on \mathcal{A}^{**} of Combes [12] and of Ortega, Rordam, and Thiel [32] .

2. PRELIMINARIES

2.1. The tracial simplex. If \mathcal{A} is a unital simple C^* -algebra of real rank zero, denote by $T(\mathcal{A})$ the collection of the tracial states on \mathcal{A} . It is well known that $T(\mathcal{A})$ is a w^* -closed convex subset of the state space of \mathcal{A} and hence is w^* -compact. $T(\mathcal{A})$ is in fact a Choquet simplex [35, Theorem 3.1.18].

When \mathcal{A} is a σ -unital but not unital simple C^* -algebra of real rank zero, denote by $\tilde{T}(\mathcal{A})$ the collection of all nonzero lower semi-continuous, semifinite tracial weights on \mathcal{A} . Recall that a weight on \mathcal{A} is a map $\tau : \mathcal{A}_+ \rightarrow [0, \infty]$ such that

- $\tau(\lambda x) = \lambda \tau(x)$ for all $x \in \mathcal{A}_+$ and all $\lambda \geq 0$.
 - $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{A}_+$.
- A weight is called tracial if furthermore
- $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{A}$;
 - it is called semifinite (or densely defined) if
 - $\{a \in \mathcal{A} \mid \tau(a^*a) < \infty\}$ is dense in \mathcal{A} ;
 - and it is called faithful if
 - $\tau(a) > 0$ for all $0 \neq a \in \mathcal{A}_+$.

In the literature, tracial weights are also called traces or extended traces, while tracial states are also called normalized traces.

Notice that lower-semicontinuous weights on a C^* -algebra of real rank zero are completely determined by their values on the projections of the algebra. Indeed, every positive element $a \in \mathcal{A}_+$ is the norm limit of a sequence of positive operators b_n with finite spectrum ([10]); in addition, the operators b_n can be chosen such that $b_n \leq a$, (e.g., see [24, Lemma 2.3]). Thus two weights that agree on projections must agree on positive operators with finite spectrum, and hence by their lower-semicontinuity, must agree also on all positive operators.

Furthermore, if $p \in \mathcal{A}$ is a nonzero projection, by the simplicity of \mathcal{A} it follows that for every projection $q \in \mathcal{A}$ there is an $n \in \mathbb{N}$ for which $[q] \leq n[p]$, where $[p]$ denotes the Murray-von Neumann equivalence class of the projection p (e.g., see [2, Corollary 6.3.6].) Thus if a lower semicontinuous tracial weight τ is finite (resp. nonzero) for one nonzero projection, then it must be finite (resp. nonzero) for all nonzero projections and hence for all positive operators with finite spectrum. Thus

if τ is nonzero, then it is faithful. Moreover, τ is semifinite if and only if $\tau(p) < \infty$ for some non-zero projection.

Notice further that every $\tau \in \tilde{T}(\mathcal{A} \otimes \mathcal{K})$ is uniquely determined by its values on $\mathcal{A}_+ \otimes e_{11}$ where $\{e_{ij}\}$ denotes the system of matrix units of the C^* -algebra of all compact operators \mathcal{K} on a separable Hilbert space. It is well known that every projection $p \in \mathcal{A} \otimes \mathcal{K}$ is unitarily equivalent to a projection in $\mathbb{M}_n(\mathcal{A}) = \mathcal{A} \otimes \mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and that any tracial weight on $(\mathcal{A} \otimes \mathbb{M}_n(\mathbb{C}))_+$ is determined by its values on $\mathcal{A}_+ \otimes e_{11}$. Thus if $\tau \in \tilde{T}(\mathcal{A})$ and Tr denotes the standard trace on \mathcal{K} , then $\tau \otimes \text{Tr}$ is the unique extension of τ to an element of $\tilde{T}(\mathcal{A} \otimes \mathcal{K})$. This permits us to identify $\tilde{T}(\mathcal{A})$ with $\tilde{T}(\mathcal{A} \otimes \mathcal{K})$.

It follows from the Brown's Stabilization Theorem [8, Corollary 2.6] that for every full projection $p \in \mathcal{M}(\mathcal{A})$ there is an isometry $w \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that

$$w(\mathcal{A} \otimes \mathcal{K})w^* = p\mathcal{A}p \otimes \mathcal{K}.$$

Thus if $\tau \in \tilde{T}(p\mathcal{A}p \otimes \mathcal{K})$, then $\tau'(\cdot) := \tau(w^* \cdot w)$ is the unique extension of τ to an element of $\tilde{T}(\mathcal{A} \otimes \mathcal{K})$.

Combining the two observations above, every tracial state τ in $T(p\mathcal{A}p)$ uniquely extends to $\tau \otimes \text{Tr} \in \tilde{T}(p\mathcal{A}p \otimes \mathcal{K})$, and in turn to $(\tau \otimes \text{Tr})(w^* \cdot w) \in \tilde{T}(\mathcal{A} \otimes \mathcal{K})$, and the restriction of $(\tau \otimes \text{Tr})(w^* \cdot w)$ to $\mathcal{A}_+ \otimes e_{11} \cong \mathcal{A}_+$ is therefore the unique extension of τ to an element of $\tilde{T}(\mathcal{A})$. Therefore, we can identify $T(p\mathcal{A}p)$ with the scaled space of $\tilde{T}(\mathcal{A})$, that is

$$(2.1) \quad T(p\mathcal{A}p) = \left\{ \frac{\tau}{\tau(p)} \mid \tau \in \tilde{T}(\mathcal{A}) \right\}.$$

The next lemma shows that the tracial simplex $T(p\mathcal{A}p)$ does not depend on the choice of the nonzero projection p .

Lemma 2.1. *Let $p, q \in \mathcal{A}$ be nonzero projections. Then there is a bijection ψ between the sets of extremal points $\text{Ext}(T(p\mathcal{A}p))$ and $\text{Ext}(T(q\mathcal{A}q))$ of $T(p\mathcal{A}p)$ and of $T(q\mathcal{A}q)$, respectively. Furthermore, if $\text{Ext}(T(p\mathcal{A}p))$ is a finite set, then the bijection ψ extends to an affine homeomorphism between $T(p\mathcal{A}p)$ and $T(q\mathcal{A}q)$.*

Proof. Define $\psi : \text{Ext}(T(p\mathcal{A}p)) \rightarrow \text{Ext}(T(q\mathcal{A}q))$ by $\psi\left(\frac{\tau}{\tau(p)}\right) = \frac{\tau}{\tau(q)}$. Then ψ is a bijection. In fact, given $\frac{\tau}{\tau(p)}$ an extreme point of $T(p\mathcal{A}p)$ for some $\tau \in \tilde{T}(\mathcal{A})$, we show that $\frac{\tau}{\tau(q)}$ is an extreme point of $T(q\mathcal{A}q)$.

We reason this by contradiction. Assume that the image $\psi\left(\frac{\tau}{\tau(p)}\right) = \frac{\tau}{\tau(q)}$ is written as the convex combination in $T(q\mathcal{A}q)$

$$\frac{\tau}{\tau(q)} = t \frac{\tau_1}{\tau_1(q)} + (1-t) \frac{\tau_2}{\tau_2(q)} \quad \text{for some } 0 < t < 1.$$

Then

$$\frac{t \tau(q) \tau_1(p)}{\tau(p) \tau_1(q)} + \frac{(1-t) \tau(q) \tau_2(p)}{\tau(p) \tau_2(q)} = 1$$

and hence

$$\begin{aligned} \frac{\tau}{\tau(p)} &= \frac{t \tau(q)}{\tau(p) \tau_1(q)} \tau_1 + \frac{(1-t) \tau(q)}{\tau(p) \tau_2(q)} \tau_2 \\ &= \frac{t \tau(q) \tau_1(p)}{\tau(p) \tau_1(q)} \frac{\tau_1}{\tau_1(p)} + \frac{(1-t) \tau(q) \tau_2(p)}{\tau(p) \tau_2(q)} \frac{\tau_2}{\tau_2(p)} \end{aligned}$$

would be a proper convex combination of tracial states $\frac{\tau_1}{\tau_1(p)}$ and $\frac{\tau_2}{\tau_2(p)}$ of $T(pAp)$, a contradiction. Thus ψ maps extreme points to extreme points. By switching the roles of p and q , we see that ψ is one-to-one and onto.

If furthermore $T(pAp)$ has only finitely many extremal points $\{\tau_j\}_1^m$, we can uniquely extend ψ to an affine map of the simplex $T(pAp)$ onto the simplex $T(qAq)$ by setting

$$\psi\left(\sum_{i=1}^m \lambda_i \tau_i\right) = \sum_{i=1}^m \lambda_i \psi(\tau_i) \quad \text{for all } 0 \leq \lambda_i \leq 1 \quad \text{with} \quad \sum_{i=1}^m \lambda_i = 1.$$

The continuity of ψ is then obvious. \square

From now on we will frequently identify $T(pAp)$ with $\tilde{T}(\mathcal{A})$ for any nonzero projection $p \in \mathcal{A}$, and denote both by $T(\mathcal{A})$ and use τ to denote both a tracial state on pAp as well as its extension to a lower semicontinuous semifinite tracial weight on \mathcal{A}_+ or on $(\mathcal{A} \otimes \mathcal{K})_+$.

2.2. Continuous affine function on the tracial simplex. Recall that $T(\mathcal{A})$ is a compact convex space. Let $\text{Aff } T(\mathcal{A})$ be the space of all real-valued, continuous, affine functions on $T(\mathcal{A})$, equipped with the uniform norm. $\text{Aff } T(\mathcal{A})$ is a closed subspace of the space of all real-valued continuous functions on $T(\mathcal{A})$, and hence is a (real) Banach space.

For every projection $p \in \mathcal{A} \otimes \mathcal{K}$ let \hat{p} denote the evaluation map

$$T(\mathcal{A}) \ni \tau \rightarrow \hat{p}(\tau) := (\tau \otimes \text{Tr})(p).$$

It is elementary to see that $\hat{p} \in \text{Aff } T(\mathcal{A})$.

Notice that if $p \in \mathcal{A}$, then of course $\hat{p}(\tau) = \tau(p)$. Notice also that if \mathcal{A} is unital and $a = a^* \in \mathcal{A}$, then the evaluation map

$$T(\mathcal{A}) \ni \tau \rightarrow \hat{a}(\tau) := \tau(a)$$

also belongs to $\text{Aff } T(\mathcal{A})$.

Now consider the map

$$(2.2) \quad \Phi : \text{Aff } T(\mathcal{A}) \ni f \rightarrow \Phi(f) = \{f(\tau)\}_{\tau \in \text{Ext}(T(\mathcal{A}))} \in \ell^\infty(\text{Ext}(T(\mathcal{A})))$$

which is clearly a linear contraction, namely $\|\Phi(f)\|_\infty \leq \|f\|$. On the other hand, for every $f \in \text{Aff } T(\mathcal{A})$

$$\sup_{\tau \in \text{co}(\text{Ext}(T(\mathcal{A})))} |f(\tau)| \leq \|\Phi(f)\|_\infty.$$

It follows from the well-known Krein-Milman theorem that

$$\|f\| = \sup_{\tau \in T(\mathcal{A})} |f(\tau)| \leq \|\Phi(f)\|_\infty.$$

Therefore, Φ is actually an isometry, namely, $\|f\| = \sup_{\tau \in T(\mathcal{A})} |f(\tau)| = \|\Phi(f)\|_\infty$.

Furthermore, when $T(\mathcal{A})$ has only finitely many extremal points, say

$$\text{Ext}(T(\mathcal{A})) = \{\tau_j\}_1^m,$$

i.e., when $T(\mathcal{A})$ is a classical simplex, then Φ is onto $\mathbb{R}^m = \ell^\infty(\text{Ext}(T(\mathcal{A})))$.

2.3. Strict comparison of projections and weak unperforation of $K_0(\mathcal{A})$.

As in the literature there is more than one definition of comparison of projections, we state explicitly below the one that we will use.

Definition 2.2. *We say that a C^* -algebra \mathcal{A} has the strict comparison of projections if $T(\mathcal{A})$ is non-empty and for any two projections p and q in \mathcal{A} , the strict inequalities $\tau(p) < \tau(q)$ for all $\tau \in T(\mathcal{A})$ imply that $p \prec q$, namely $[p] < [q]$, the strict ordering induced by the Murray-von Neumann equivalence of projections.*

The above definition was given in [4, FCQ2,1.3.1] for simple C^* -algebras, while we only work with simple C^* -algebras of real rank zero. For ease of reference, let us state without proof the following simple observation.

Lemma 2.3. *Let \mathcal{A} be a simple, unital C^* -algebra of real rank zero such that \mathcal{A} has the strict comparison of projections. If $p \in \mathcal{A} \otimes \mathcal{K}$ is a projection such that $(\tau \otimes \text{Tr})(p) < 1$ for all $\tau \in T(\mathcal{A})$, then there is a projection $p' \in \mathcal{A}$ such that $p' \otimes e_{11} \sim p$ in $\mathcal{A} \otimes \mathcal{K}$. In particular, $(\tau \otimes \text{Tr})(p) = \tau(p')$ for all $\tau \in T(\mathcal{A})$.*

As a trivial fact, if $\mathcal{A} \otimes \mathcal{K}$ has the strict comparison of projections, then so does \mathcal{A} . Less obvious is the converse as follows.

Lemma 2.4. *Let \mathcal{A} be a σ -unital simple C^* -algebra of real rank zero. Then the following are equivalent:*

- (i) \mathcal{A} has the strict comparison of projections.
- (ii) $M_n(\mathcal{A})$ has the strict comparison of projections for every $n \in \mathbb{N}$.
- (iii) $\mathcal{A} \otimes \mathcal{K}$ has the strict comparison of projections.

Proof. The implications (iii) \Rightarrow (ii) \Rightarrow (i) are obvious. We need only to prove that (i) implies (iii).

To simplify notations, denote by $\tau \in T(\mathcal{A})$ both a tracial weight on $\mathcal{A} \otimes \mathcal{K}$ as well as its restriction to \mathcal{A} . Let p, q be projections in $\mathcal{A} \otimes \mathcal{K}$, such that $\tau(p) < \tau(q)$ for all $\tau \in T(\mathcal{A})$. Take an approximate identity $\{e_\lambda \mid \lambda \in \Lambda\}$ of \mathcal{A} . Then $\{\sum_{i=1}^n e_\lambda \otimes e_{ii} \mid (\lambda, n) \in \Lambda \times \mathbb{N}\}$ is an approximate identity of $\mathcal{A} \otimes \mathcal{K}$. For $p, q \in \mathcal{A} \otimes \mathcal{K}$, a standard argument shows that p, q are equivalent to projections p', q' in $e_{\lambda_0} \mathcal{A} e_{\lambda_0} \otimes \mathcal{K}$ for some λ_0 . Therefore, from now on we can assume that \mathcal{A} is unital.

Recall that the evaluation map $\hat{q} - \hat{p}$ belongs to $\text{Aff } T(\mathcal{A})$ and choose

$$0 < \delta < \min \left(\inf_{\tau \in T(\mathcal{A})} (\tau(q) - \tau(p)), 2 \right).$$

Applying the diagonalization of projections in \mathcal{A} or in $\mathcal{A} \otimes \mathcal{K}$, respectively, proved in [42, 1.1,1.4] and [41, 1.2], one can find for every $k \in \mathbb{N}$ projections p_k, q_k, r_k, r'_k in $\mathcal{A} \otimes \mathcal{K}$ such that

$$\begin{aligned} [p] &= 2^k [p_k] + [r_k], \quad 0 < [r_k] < [p_k] \\ [q] &= 2^k [q_k] + [r'_k], \quad 0 < [r'_k] < [q_k]. \end{aligned}$$

By iterating the diagonalization above to r_k one can also assume that $[r_k] < [r'_k]$.

Choose k such that $\frac{1}{2^k} \sup_{\tau \in T(\mathcal{A})} \tau(p) < \frac{\delta}{2}$ and $\frac{1}{2^k} \sup_{\tau \in T(\mathcal{A})} \tau(q) < \frac{\delta}{2}$. In particular, $\tau(p_k) \leq \frac{\delta}{2} < 1$ and $\tau(q_k) \leq \frac{\delta}{2} < 1$ and the same holds for r_k and r'_k . By Lemma 2.3, p_k, q_k, r_k , and r'_k are equivalent to projections in \mathcal{A} . Thus, without loss of generality, we can assume $p_k, q_k, r_k, r'_k \in \mathcal{A}$. Since $[r_k] < [r'_k]$, it follows that $\tau(r_k) < \tau(r'_k)$. Applying the strict comparison of projections of \mathcal{A} , one sees $r_k \precsim r'_k$ in \mathcal{A} .

Moreover, for every $\tau \in T(\mathcal{A})$,

$$\tau(q) - \tau(p) = 2^k \tau(q_k) + \tau(r'_k) - 2^k \tau(p_k) - \tau(r_k) > \delta$$

hence

$$2^k (\tau(q_k) - \tau(p_k)) > \delta - \tau(r'_k) > \frac{\delta}{2} > 0.$$

Thus $\tau(q_k) > \tau(p_k)$ for every $\tau \in T(\mathcal{A})$. By the strict comparison of projections of \mathcal{A} , it follows that $p_k \precsim q_k$ in \mathcal{A} . But then $p \precsim q$ in $\mathcal{A} \otimes \mathcal{K}$, which concludes the proof. \square

Finally, we notice the following direct consequence.

Remark 2.5. *Let \mathcal{A} be a simple σ -unital C^* -algebra with real rank zero. If $T(\mathcal{A})$ has the strict comparison of projections, then $K_0(\mathcal{A})$ is weakly unperforated.*

Proof. We sketch the reasoning only for the reader's convenience. For any $x = [p] - [q] \in K_0(\mathcal{A})$ with $nx > 0$ for some $n \in \mathbb{N}$, where projections $p, q \in \mathcal{A} \otimes \mathcal{K}$. Then $nx = [r]$ for some projection $0 \neq r \in \mathcal{A} \otimes \mathcal{K}$ by the strict comparison of projections. We show that $x > 0$. In fact, $n[p] = n[q] + [r]$ and for all $\tau \in T(\mathcal{A})$ (identified with $T(\mathcal{A} \otimes \mathcal{K})$). It follows that $n\tau(p) = n\tau(q) + \tau(r)$ and hence $\tau(p) = \tau(q) + \frac{1}{n}\tau(r) > \tau(q)$. By the strict comparison of projections of \mathcal{A} , $q \precsim p$ and hence $x > 0$. \square

Remark 2.6. *For any unital C^* -algebra the property of having stable rank one implies cancellation ([2, 6.5.1]). Assume further that a C^* -algebra has real rank zero. Then the cancellation property is equivalent to stable rank one ([2, 6.5.2]).*

2.4. Quasitraces. Recall that a quasitrace τ on a C^* -algebra \mathcal{A} satisfies the same properties of a trace with exception of additivity, and that $\tau(a+b) = \tau(a) + \tau(b)$ for $a, b \in \mathcal{A}_{sa}$ under the additional hypothesis that a and b commute. A 2-quasitrace on \mathcal{A} is a quasitrace that has a quasitrace extension to $\mathcal{A} \otimes \mathbb{M}_2(\mathbb{C})$ and hence extends to $\mathbb{M}_n(\mathbb{C})$ for every $n \in \mathbb{N}$ and thus to $\mathcal{A} \otimes \mathcal{K}$. $QT(\mathcal{A})$ denotes the collection of 2-quasitraces on \mathcal{A} . $QT(\mathcal{A})$ too is a Choquet simplex and contains $T(\mathcal{A})$ as closed face [5, Proposition II 4.5].

Notice that for every projection $p \in \mathcal{A} \otimes \mathcal{K}$, the evaluation map

$$QT(\mathcal{A}) \ni \tau \rightarrow \tau(p)$$

also belongs to $\text{Aff } QT(\mathcal{A})$ and we will still denote it by \hat{p} .

We will use the following density property [2, Theorem 6.9.3] (see also [5, Lemma III.3.4]): if \mathcal{A} is simple, unital, non-elementary (i.e., $\mathcal{A} \otimes \mathcal{K} \not\cong \mathcal{K}$), stably finite, of real rank zero, stable rank one, and has weakly unperforated $K_0(\mathcal{A})$, then $\{\hat{x} \mid x \in K_0(\mathcal{A})\}$ is uniformly dense in $\text{Aff } QT(\mathcal{A})$, namely, for every $f \in \text{Aff } QT(\mathcal{A})$ and every $\epsilon > 0$ there is an element $x \in K_0(\mathcal{A})$ such that

$$(2.3) \quad |\tau(x) - f(\tau)| < \epsilon \quad \forall \tau \in QT(\mathcal{A}).$$

It is convenient to mention explicitly the following consequence of the above density property.

Remark 2.7. *If \mathcal{A} is simple, unital, non-elementary, stably finite, of real rank zero, stable rank one, with weakly unperforated $K_0(\mathcal{A})$, and with $T(\mathcal{A}) \neq \emptyset$, then the semigroup $D(\mathcal{A} \otimes \mathcal{K})$ of all equivalence classes of projections in $\mathcal{A} \otimes \mathcal{K}$, which we*

identify with $\{\hat{p} \mid [p] \in D(\mathcal{A} \otimes \mathcal{K})\}$, is dense in $\text{Aff } T(\mathcal{A})_+$; equivalently, for every $f \in \text{Aff } T(\mathcal{A})_+$ and every $\epsilon > 0$ there is a projection $p \in \mathcal{A} \otimes \mathcal{K}$ such that

$$(2.4) \quad |\tau(p) - f(\tau)| < \epsilon \quad \forall \tau \in T(\mathcal{A}).$$

Proof. It is enough to notice that $T(\mathcal{A})$ is a closed face of $QT(\mathcal{A})$ [5, Proposition II 4.5] and $QT(\mathcal{A})$ is a Choquet simplex. Then every $f \in \text{Aff } T(\mathcal{A})$ has an extension to an $\tilde{f} \in \text{Aff } QT(\mathcal{A})$ such that $\inf_{\tau \in T(\mathcal{A})} f(\tau) \leq \tilde{f} \leq \sup_{\tau \in T(\mathcal{A})} f(\tau)$ [22, Theorem 11.22]. In particular, if $f \in \text{Aff } T(\mathcal{A})_+$, then $\tilde{f} \in \text{Aff } QT(\mathcal{A})_+$. But then the same projection $p \in \mathcal{A} \otimes \mathcal{K}$ that satisfies (2.3), satisfies a fortiori the condition in (2.4). \square

A celebrated (unpublished) result of Haagerup [23] states that if \mathcal{A} is unital and exact, then $QT(\mathcal{A}) = T(\mathcal{A})$. Then Blanchard and Kirchberg [6, Remark 2.29 (i)] observed that this result can be extended non-unital exact C^* -algebras. Brown and Winter [11] provided a short proof of Haagerup's result in the finite nuclear dimension case.

We show now that $QT(\mathcal{A}) = T(\mathcal{A})$ holds also for the C^* -algebras considered in the present paper. First, we recall the following fact which is an immediate consequence of [22, Theorem 11.22].

Lemma 2.8. *Let K be a Choquet simplex, let $F \subset K$ be a closed face of K , and let $x \in \text{Ext}(K) \setminus F$. Then for all $\alpha, \beta \in \mathbb{R}_+$ there is a $g \in \text{Aff}(K)_+$ such that $g|_F = \alpha$ and $g(x) = \beta$.*

Theorem 2.9. *If \mathcal{A} is a unital simple, C^* -algebra of real rank zero, stable rank one, and has the strict comparison of projections, then $QT(\mathcal{A}) = T(\mathcal{A})$.*

Proof. The case when \mathcal{A} is elementary algebra is trivial. We can assume henceforth that \mathcal{A} is non-elementary. Reasoning by contradiction, assume that $T(\mathcal{A})$ is a proper subset of $QT(\mathcal{A})$. Then there is an extreme point τ_o of $QT(\mathcal{A})$ but not in $T(\mathcal{A})$. Then $\{\tau_o\}$ and $T(\mathcal{A})$ are closed disjoint faces of $QT(\mathcal{A})$. Notice that constant functions on a face are continuous and affine.

Thus by Lemma 2.8, there are positive continuous affine functions $f, g \in \text{Aff } QT(\mathcal{A})$ such that

$$\begin{cases} f|_{T(\mathcal{A})} = \frac{1}{2} & f(\tau_o) = 0 \\ g|_{T(\mathcal{A})} = 0 & g(\tau_o) = \frac{1}{2}. \end{cases}$$

By Remark 2.5, \mathcal{A} is also weakly unperforated, hence the conditions for the density of the projections in $\text{Aff } QT(\mathcal{A})_+$ (see (2.3)) are satisfied. Thus there are projections $p, q \in \mathcal{A} \otimes \mathcal{K}$ such that

$$\begin{cases} \sup_{\tau \in QT(\mathcal{A})} |\tau(p) - f(\tau)| < \frac{1}{4} \\ \sup_{\tau \in QT(\mathcal{A})} |\tau(q) - g(\tau)| < \frac{1}{4}. \end{cases}$$

In particular,

$$\tau(q) < \frac{1}{4} < \tau(p) \quad \forall \tau \in T(\mathcal{A}).$$

By Lemma 2.3 there are projections $p', q' \in \mathcal{A}$ such that $p' \sim p$, $q' \sim q$. Thus assume without loss of generality that $p, q \in \mathcal{A}$. Then by the strict comparison of projections, $q \precsim p$. This implies $\tau_o(q) \leq \tau_o(p)$, whereas $\tau_o(p) < \frac{1}{4} < \tau_o(q)$, a contradiction. \square

3. SUMS OF COMMUTATORS

We start with the following simple extension of a result by Thomsen on uniform algebras.

Lemma 3.1. *Let $\mathcal{C} = \bigoplus_{i=1}^N p_i M_{n_i}(C(X_i)) p_i$ where for each i , $n_i \in \mathbb{N}$, X_i is a compact Hausdorff space with covering dimension $d_i \leq d$, and $p_i \in \mathbb{M}_{n_i}(C(X_i))$ is a nonzero projection. Let $a \in \mathcal{A}$ be a self-adjoint element, let $\eta > 0$, and assume that $|\tau(a)| \leq \eta$ for all $\tau \in T(\mathcal{C})$. Then for every $\epsilon > 0$ there exist $v_1, v_2, \dots, v_d \in \mathcal{C}$ such that $\|v_i\| \leq \sqrt{2}\|a\|^{1/2}$ for $1 \leq i \leq d$ and*

$$\|a - \sum_{i=1}^{d+1} [v_i, v_i^*]\| < \eta + \epsilon.$$

Proof. If $\tau \in T(\mathcal{C})$ and $\tau(p_i) \neq 0$, the restriction of $\frac{\tau}{\tau(p_i)}$ to $p_i M_{n_i}(C(X_i)) p_i$ is a tracial state on $p_i M_{n_i}(C(X_i)) p_i$. Since $\sum_{i=1}^N \tau(p_i) = 1$, $T(\mathcal{C})$ is the collection of the convex combinations of the elements of $T(p_i M_{n_i}(C(X_i)) p_i)$. Thus it is enough to prove the statement for the case that $N = 1$, i.e. when $\mathcal{C} = p M_n(C(X)) p$.

Since the range of the continuous function $\text{Tr}(p(x))$ consists of integers, it follows that $X_0 := \{x \in X \mid \text{Tr}(p(x)) = 0\}$ is a closed connected component of X and

$$p M_n(C(X)) p = p M_n(C(X \setminus X_0)) p.$$

Thus we can assume without loss of generality that $p(x) \neq 0$ for all $x \in X$ and hence $1 \leq \text{Tr}(p(x)) \leq n$ for all $x \in X$. For every extremal $\tau \in T(\mathcal{C})$, there is an $x \in X$ for which $\tau(c) = \frac{\text{Tr}(c p(x))}{\text{Tr}(p(x))}$ for every $c \in \mathcal{C}$. Define $f(x) = \frac{\text{Tr}(a p(x))}{\text{Tr}(p(x))}$, then $f \in C(X)$ and $|f(x)| \leq \eta$ for every $x \in X$. Define $b := fp$. Then $b \in \mathcal{C}$, $\|b\| \leq \eta$, and

$$\text{Tr}(b(x)) = \text{Tr}(f(x)p(x)) = f(x) \text{Tr}(p(x)) = \text{Tr}(a(x)).$$

Thus $\text{Tr}((a-b)(x)) = 0$ for every $x \in X$ and hence it follows from Thomsen's result [36, Lemma 1.4] that there exist $v_1, v_2, \dots, v_d \in p M_n(C(X)) p$ such that $\|v_i\| \leq \sqrt{2}\|a\|^{1/2}$ for $1 \leq i \leq d$ and $\|a - b - \sum_{i=1}^{d+1} [v_i, v_i^*]\| < \epsilon$. As a consequence, $\|a - \sum_{i=1}^{d+1} [v_i, v_i^*]\| < \eta + \epsilon$. \square

Lemma 3.2. *Let \mathcal{C} be the C^* -inductive limit $\mathcal{C} = \lim_{n \rightarrow \infty} (\mathcal{C}_n, \phi_{n,n+1})$ of a sequence of unital C^* -algebras $\{\mathcal{C}_n\}$ such that $T(\mathcal{C}_n) \neq \emptyset$ for all n and the connecting maps $\phi_{n,n+1} : \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ are unital and one-to-one.*

- (i) *Let $n \in \mathbb{N}$ and $a \in \mathcal{C}_n$ be a self-adjoint element such that $\tau(a) > 0$ for all $\tau \in T(\mathcal{C})$. Then there exists an $M \geq n$ such that $\tau(\phi_{n,m}(a)) > 0$ for all integers $m \geq M$ and for all $\tau \in T(\mathcal{C}_m)$.*
- (ii) *Let $\epsilon > 0$, $n \in \mathbb{N}$, $a \in \mathcal{C}_n$ be a self-adjoint element such that $|\tau(a)| < \epsilon$ for all $\tau \in T(\mathcal{C})$. Then there exists $M \geq n$ such that $|\tau(\phi_{n,m}(a))| < \epsilon$ for all $\tau \in T(\mathcal{C}_m)$ and for all $m \geq M$.*

Proof. To simplify notations, assume without loss of generality that the C^* -algebras \mathcal{C}_n form a nested sequence of subalgebras of \mathcal{C} for which $\mathcal{C} = \overline{\bigcup_n \mathcal{C}_n}$ and thus the connecting map $\phi_{n,n+1}$ is just the identity embedding of \mathcal{C}_n into \mathcal{C}_{n+1} .

- (i) Suppose, to the contrary, that for some increasing sequence of integers $n_k \geq n$ there is a sequence $\tau_k \in T(\mathcal{C}_{n_k})$ for which

$$(3.1) \quad \tau_k(a) \leq 0.$$

Let $\tilde{\tau}_k$ be an arbitrary extension of τ_k to a state of \mathcal{C} . The state space of \mathcal{C} being w^* -compact, there is some w^* -converging subnet of the sequence $\tilde{\tau}_k$. To simplify notations, we can assume that $\tilde{\tau}_k \rightarrow \tau$ where τ is a state of \mathcal{C} .

We claim that τ is a trace. For every $a \in \mathcal{C}$, let $a_n \in \mathcal{C}_n$ be a sequence converging to a . Then

$$\begin{aligned}
\tau(aa^*) &= \lim_n \tau(a_n a_n^*) && \text{(by the continuity of } \tau) \\
&= \lim_n \lim_k \tilde{\tau}_k(a_n a_n^*) && \text{(by the definition of } \tau) \\
&= \lim_n \lim_k \tau_k(a_n a_n^*) && \text{(because } a_n \in \mathcal{C}_{n_k} \text{ for all } n_k \geq n) \\
&= \lim_n \lim_k \tau_k(a_n^* a_n) && \text{(because } \tau_k \text{ is a trace on } \mathcal{C}_{n_k}) \\
&= \tau(a^* a) && \text{(reversing the above argument.)}
\end{aligned}$$

From the definition of τ and from (3.1), we have that $\tau(a) \leq 0$, a contradiction.

(ii) For every $\tau \in T(\mathcal{C})$, $\tau(\epsilon I - a) > 0$ and $\tau(\epsilon I + a) > 0$. Thus by (i), there is an $M_1 \geq n$ (resp. $M_2 \geq n$) such that $\tau(\epsilon I - a) > 0$ for every $\tau \in T(A_m)$ and every $m \geq M_1$, (resp. $\tau(\epsilon I + a) > 0$ for every $\tau \in T(A_m)$ and every $m \geq M_2$). The conclusion then follows by taking $M = \max\{M_1, M_2\}$. \square

From [13, Theorems 2.9, 3.4] we know that every selfadjoint element in the kernel of all tracial states of a simple unital C^* -algebra is the norm limit of sums of selfcommutators. We need however to obtain a bound on the number of the commutators and on the norms of the elements in the commutators. This was obtained for unital simple AH-algebra \mathcal{C} with real rank zero and bounded dimension growth and for some other algebras in [29]. Based on the work in [27], we can extend these results to a larger class of C^* -algebras. We start with an approximation property for elements with uniformly bounded traces, which include of course those in the kernel of all traces.

Lemma 3.3. *Let \mathcal{A} be a unital separable simple C^* -algebra of real rank zero, stable rank one, and the strict comparison of projections. Let $a \in \mathcal{A}$ be a self-adjoint element, let $\eta > 0$, and assume that $|\tau(a)| \leq \eta$ for all $\tau \in T(\mathcal{A})$. Then for every $\epsilon > 0$ there exist $v_1, v_2, v_3, v_4 \in \mathcal{A}$ such that $\|v_i\| \leq \sqrt{2}\|a\|^{1/2}$ for $1 \leq i \leq 4$ and*

$$\|a - \sum_{i=1}^4 [v_i, v_i^*]\| < \eta + \epsilon.$$

Proof. Assume without loss of generality that $\|a\| = 1$. Since \mathcal{A} has real rank zero, there is a selfadjoint element of finite spectrum a' with $\|a' - (1 - \frac{\epsilon}{6})a\| < \frac{\epsilon}{6}$. Thus $\|a'\| \leq 1$ and $\|a' - a\| < \frac{\epsilon}{3}$.

Notice that for every $\tau \in T(\mathcal{A})$ we have

$$|\tau(a')| \leq \eta + |\tau(a) - \tau(a')| \leq \eta + \|a - a'\| < \eta + \frac{\epsilon}{3}.$$

Write $a' = \sum_1^k \lambda_j p_j$ for some mutually orthogonal projections $p_j \in \mathcal{A}$ and $\lambda_j \in \mathbb{R}$.

By [27, Theorem 4.5], which, as pointed out by Emmanuel C. Germain in the review MR1869626 (2002i:46053), holds also when \mathcal{A} is not necessarily nuclear, (see also [14, Theorem 4.20] and [15]), there exists a unital simple AH-algebra \mathcal{C} with real rank zero and dimension bounded by three, and a unital $*$ -embedding

$\Psi : \mathcal{C} \rightarrow \mathcal{A}$ such that Ψ induces an isomorphism of the K-theory invariant:

$$K_*(\Psi) : (K_0(\mathcal{C}), K_0(\mathcal{C})_+, K_1(\mathcal{C}), [1_{\mathcal{C}}]) \rightarrow (K_0(\mathcal{A}), K_0(\mathcal{A})_+, K_1(\mathcal{A}), [1_{\mathcal{A}}]).$$

Notice that the induced map on tracial simplexes, $T(\Psi) : T(\mathcal{A}) \rightarrow T(\mathcal{C})$ is an affine homeomorphism of compact convex sets (see [2, Theorem 6.9.1]) and all the tracial states of $T(\mathcal{C})$ extends to tracial states on \mathcal{A} . To simplify notations, assume that \mathcal{C} is a subalgebra of \mathcal{A} sharing the unit with \mathcal{A} and that Ψ is the natural inclusion map.

Decompose \mathcal{C} into a C*-inductive limit $\mathcal{C} = \lim_{n \rightarrow \infty} (\mathcal{C}_n, \phi_{n,n+1})$ where each connecting map $\phi_{n,n+1} : \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ is unital and injective, and where each \mathcal{C}_n is a finite direct sum of unital homogeneous C*-algebras with spectrum being a path-connected finite CW-complex with dimension less than or equal to three. Denote by ϕ_n the unital map embedding \mathcal{C}_n in $\mathcal{C} = \cup_n \phi_n(\mathcal{C}_n)$ and hence in \mathcal{A} .

Notice that by the assumption of stable rank one and hence cancellation, two Murray-von Neumann equivalent projections in \mathcal{A} are necessarily unitarily equivalent. Moreover, by the isomorphism of the ordered K-groups, every projection $p \in \mathcal{A}$ is equivalent to a projection $q \in \mathcal{C}$ and furthermore, every projection in \mathcal{C} is unitarily equivalent to a projection in $\phi_n(\mathcal{C}_n)$ for some $n \in \mathbb{N}$ (e.g., see [39, Proposition 6.2.9, Corollary 5.1.7, and Appendix L. 2.2].) Thus, by considering $p := \sum_1^k p_j$ we can find a unitary $u \in \mathcal{A}$ for which $up_j u^* \in \phi_n(\mathcal{C}_n)$ for all $1 \leq j \leq k$. Let $q_j := \phi_n^{-1}(up_j u^*)$ and let $a'' := \sum_1^k \lambda_j q_j$. Since $a' = u\phi_n(a'')u^*$,

$$|\tau(\phi_n(a''))| = |\tau(a')| < \eta + \frac{\epsilon}{3} \quad \text{for every } \tau \in T(\mathcal{A})$$

and hence

$$|\tau(\phi_n(a''))| < \eta + \frac{\epsilon}{3} \quad \text{for every } \tau \in T(\mathcal{C}).$$

By Lemma 3.2 there is some $m \geq n$ (actually, for every $m' \geq m$) for which

$$|\tau(a'')| < \eta + \frac{\epsilon}{3} \quad \text{for every } \tau \in T(\mathcal{C}_m).$$

But then by Lemma 3.1, there are four elements $v_i \in \mathcal{C}_m$ with

$$\|v_i\| \leq \sqrt{2}\|a''\|^{\frac{1}{2}} = \sqrt{2}\|a'\|^{\frac{1}{2}} \leq \sqrt{2}\|a\|^{\frac{1}{2}}$$

and such that $\|a'' - \sum_1^4 [v_i, v_i^*]\| \leq \eta + \frac{2\epsilon}{3}$. Thus

$$\|a' - u \sum_1^4 [\phi_m(v_i), \phi_m(v_i^*)]u^*\| \leq \eta + \frac{2\epsilon}{3}$$

and hence

$$\|a - u \sum_1^4 [\phi_m(v_i), \phi_m(v_i^*)]u^*\| \leq \eta + \epsilon.$$

□

The reduction argument in [29] shows that every element in the kernel of the unique tracial state is the sum of two commutators and every selfadjoint element is the sum of four selfcommutators. The same reduction provides the following result in our setting.

Theorem 3.4. *Let \mathcal{A} be a unital simple separable C^* -algebra with real rank zero, stable rank one, and strict comparison of projections. Let $a \in \mathcal{A}$ be an element such that $\tau(a) = 0$ for all $\tau \in T(\mathcal{A})$. Then a is the sum of two commutators; i.e., there exist $y_1, y_2, y_3, y_4 \in \mathcal{A}$ such that*

$$a = [y_1, y_2] + [y_3, y_4].$$

If, in addition, a is self-adjoint then a can be expressed as the sum of four self-commutators; i.e., there exist $x_1, x_2, x_3, x_4 \in \mathcal{A}$ such that

$$a = \sum_{i=1}^4 [x_i, x_i^*].$$

Proof. The first step is to show that if $a \in \mathcal{A}$ is selfadjoint, then there are twelve elements $x_1, x_2, \dots, x_{12} \in \mathcal{A}$ with $\|x_i\| \leq 13\|a\|^{1/2}$ such that

$$a = \sum_{i=1}^{12} [x_i, x_i^*]$$

The proof is essentially the same as that of [29] 3.9, which Marcoux presents as an adaptation of T. Fack proof of [16, Theorem 3.1] and its modification by K. Thomsen [36, Theorem 1.8]. We just have to replace the key step in Marcoux's proof, namely [29, Proposition 3.6] obtained in the case of a unique tracial state, with Lemma 3.3 that we have obtained above. Since the former approximates selfadjoint elements by the sum of two selfcommutators while the latter needs four, the same proof now decomposes a into a sum of 12 selfcommutators.

The further reduction to two commutators or in the selfadjoint case to four self-commutators follows then from [29, Theorem 3.10]. \square

While Marcoux's theorems on which we depend do not present explicitly norm bounds for the elements composing the commutators, these bounds are implicit in his proofs and are further quoted explicitly in his Remark 5.3 in [29]. We do not need a value for these bounds as their existence suffices for our needs.

Remark 3.5. *There is a constant M , independent of the algebra \mathcal{A} and of the element $a \in \mathcal{A}$, such that the elements $y_i, x_i \in \mathcal{A}$ in the above theorem satisfy $\|y_i\| \leq M\|a\|^{1/2}$ and $\|x_i\| \leq M\|a\|^{1/2}$ for $1 \leq i \leq 4$.*

As a corollary, we get the following (see [30] Section 2):

Corollary 3.6. *Let \mathcal{A} be a unital separable simple C^* -algebra with real rank zero, stable rank one, and strict comparison of projections. Then $[\mathcal{A}, \mathcal{A}]$ (the linear span of the commutators of \mathcal{A}) is norm-closed.*

4. LINEAR COMBINATION OF PROJECTIONS

In the previous section, we have seen that elements belonging to the kernel of all tracial states are sums of commutators. As shown by Marcoux in [28, Theorem 3.8], under mild conditions every commutator is a linear combination of projections. In fact, implicit in his proof and in the proof of his preceding lemmas is also an estimate on the number of projections needed and on the coefficients in that linear combination. Such an estimate is stated in [30, Theorems 3.1, 3.3, 3.4]. A discussion of that estimate is also given in our previous paper [25].

Lemma 4.1. [25, Lemma 2.4] *Let \mathcal{A} be a unital C^* -algebra for which there exist three mutually orthogonal projections p_1, p_2 and p_3 such that $I = p_1 + p_2 + p_3$ and $p_i \precsim I - p_i$ for $1 \leq i \leq 3$. Then for all $x, y \in \mathcal{A}$ with $\|x\|, \|y\| \leq 1$, there exist $n \leq 84$ projections $q_1, q_2, \dots, q_n \in \mathcal{A}$ and real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $|\alpha_j| \leq 2\sqrt{2}$ such that:*

$$[x, y] := xy - yx = \alpha_1 q_1 + \alpha_2 q_2 + \dots + \alpha_n q_n.$$

Note that such projections p_1, p_2, p_3 above exist in every unital simple real rank zero C^* -algebra of dimension at least 3 by [42, Theorem 1.1]

Thus combining Theorem 3.4 and 4.1 we obtain the following result.

Theorem 4.2. *Let \mathcal{A} be a unital simple separable C^* -algebra with real rank zero, stable rank one, and the strict comparison of projections. Then every element $a \in \mathcal{A}$ such that $\tau(a) = 0$ for all $\tau \in T(\mathcal{A})$ is a linear combination $\sum_{j=1}^{168} \alpha_j p_j$ of projections $p_j \in \mathcal{A}$ with $|\alpha_j| \leq 2\sqrt{2}M^2\|a\|$ where M is the constant referred to in Remark 3.5.*

If \mathcal{A} is unital and has a unique tracial state τ , then every $a \in \mathcal{A}$ has natural decomposition

$$a = \tau(a)I + (a - \tau(a)I)$$

into a scalar multiple of a projection and an element belonging to the kernel of the trace. Under our additional hypotheses on \mathcal{A} , the same holds also in the case when $T(\mathcal{A})$ has only finitely many extremal points. Furthermore, we can control the coefficients in the linear combination of projections.

Lemma 4.3. *Let \mathcal{A} be a simple unital C^* -algebra of real rank zero having strict comparison of projections and assume that $T(\mathcal{A})$ has extremal points $\{\tau_1, \tau_2, \dots, \tau_m\}$. Then for every $\nu > 0$ and every $a \in \mathcal{A}_{sa}$ there exist real numbers $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ and projections $\{p_1, p_2, \dots, p_m\}$ in \mathcal{A} such that $\tau(a - \sum_{j=1}^m \lambda_j p_j) = 0$ for every $\tau \in T(\mathcal{A})$ and $\sum_{j=1}^m |\lambda_j| \leq (m + \nu)\|a\|$.*

Proof. The case when \mathcal{A} is a matrix algebra being trivial, assume without loss of generality that \mathcal{A} is not elementary. Let

$$\Phi : \text{Aff } T(\mathcal{A}) \ni f \rightarrow \Phi(f) = \{f(\tau)\}_{\tau \in \text{Ext}(T(\mathcal{A}))} \in \ell^\infty(\text{Ext}(T(\mathcal{A}))) = \mathbb{R}^m$$

be the linear isometry defined in (2.2) and let e_j be the standard basis of \mathbb{R}^m . Then $f_j := \Phi^{-1}(e_j)$ is a basis of $\text{Aff } T(\mathcal{A})$.

Notice that $f_j(\tau_i) = \delta_{i,j} \geq 0$ for every $1 \leq i, j \leq m$, thus $f_j(\tau) \geq 0$ for every $\tau \in T(\mathcal{A})$ and every j . Thus by Remark 2.7 for every $0 < \delta < 1$ there exist projections $p_j \in \mathcal{A} \otimes \mathcal{K}$ such that $\|\hat{p}_j - (1 - \delta)f_j\| \leq \frac{\delta}{2}$ for every j . As a consequence, $\|\Phi(\hat{p}_j) - e_j\|_\infty < \frac{3\delta}{2}$. Thus

$$\tau(p_j) \leq (1 - \delta)f_j(\tau) + \frac{\delta}{2} \leq 1 - \frac{\delta}{2} \quad \text{for every } \tau \in T(\mathcal{A}) \text{ and for every } j.$$

By Lemma 2.3, there are projections in \mathcal{A} with the same traces as p_j , thus we can assume without loss of generality that $p_j \in \mathcal{A}$.

Let B be the $m \times m$ matrix with columns $\Phi(\hat{p}_j)$. We can choose δ small enough so that $\|B - I\| \leq \frac{\nu}{2m^{3/2}}$. Reducing if necessary ν , we see that B is invertible and $\|B^{-1} - I\| \leq \frac{\nu}{m^{3/2}}$. Thus $\{\Phi(\hat{p}_j)\}$ is a basis of \mathbb{R}^m and hence $\{\hat{p}_j\}$ is a basis of

$\text{Aff } T(\mathcal{A})$. Thus, given a selfadjoint $a \in \mathcal{A}_{sa}$, there are (unique) scalars $\lambda_j \in \mathbb{R}$ so that $\hat{a} = \sum_1^m \lambda_j \hat{p}_j$, that is

$$\tau(a) = \sum_1^m \lambda_j \tau(p_j) \quad \forall \tau \in T(\mathcal{A}).$$

Set $\lambda := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$. Hence

$$\Phi(\hat{a}) = \sum_1^m \lambda_j \Phi(\hat{p}_j) = B\lambda \quad \text{and thus} \quad \lambda = B^{-1}\Phi(\hat{a}).$$

Thus

$$\begin{aligned} \sum_1^m |\lambda_j| &\leq m\|\lambda\|_\infty \\ &= m\|B^{-1}\Phi(\hat{a})\|_\infty \\ &\leq m\|\Phi(\hat{a})\|_\infty + m\|(B^{-1} - I)\Phi(\hat{a})\|_\infty \\ &\leq m\|\Phi(\hat{a})\|_\infty + m\|(B^{-1} - I)\Phi(\hat{a})\|_2 \\ &\leq m\|\Phi(\hat{a})\|_\infty + m\|B^{-1} - I\| \|\Phi(\hat{a})\|_2 \\ &\leq m\|\Phi(\hat{a})\|_\infty + m^{3/2}\|B^{-1} - I\| \|\Phi(\hat{a})\|_\infty \\ &\leq (m + \nu)\|\Phi(\hat{a})\|_\infty \\ &= (m + \nu)\|\hat{a}\| \\ &\leq (m + \nu)\|a\|. \end{aligned}$$

□

By combining Theorem 4.2 and Lemma 4.3 we thus obtain that the C^* -algebra \mathcal{A} is the linear span of its projections and furthermore has a universal constant V_o as in the following theorem.

Theorem 4.4. *Let \mathcal{A} be a unital simple separable C^* -algebra with real rank zero, stable rank one, strict comparison of projections and assume that $T(\mathcal{A})$ has a finite number of extremal points. Then \mathcal{A} is the linear span of its projections. Furthermore, there exists a positive integer $N \geq 1$ and a positive constant $V_0 > 0$ such that for every $a \in \mathcal{A}$ there exist an integer $n \leq N$, complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $|\alpha_i| \leq V_0\|a\|$ for $1 \leq i \leq n$, and projections $p_1, p_2, \dots, p_n \in \mathcal{A}$ such that $a = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$.*

5. C^* -ALGEBRAS THAT ARE NOT THE SPAN OF THEIR PROJECTIONS

5.1. Infinitely many extremal traces. If we relax the condition that $T(\mathcal{A})$ has finitely many extremal points, \mathcal{A} may fail to be the span of its projections.

Proposition 5.1. *Let \mathcal{A} be a simple σ -unital C^* -algebra of real rank zero such that $\text{Ext}(T(\mathcal{A}))$ is infinite, the collection $D(\mathcal{A})$ of Murray-von Neumann equivalence classes of projections of \mathcal{A} is countable, and $D(\mathcal{A} \otimes \mathcal{K})$ is dense in $\text{Aff } T(\mathcal{A})_+$ (see Remark 2.7). Then \mathcal{A} is not the linear span of its projections.*

Proof. Let $\{[p_j]\}$ be an enumeration of $D(\mathcal{A})$ and let $x_j := \hat{p}_j$. Recall that x_j does not depend on the representative projection p_j , that $x_j \in \text{Aff } T(\mathcal{A})$, and that $\|x_j\| \leq 1$ for all j .

Recall that by [43] every projection $p \in \mathcal{A} \otimes \mathcal{K}$ is unitarily equivalent to the direct sum of a finite number of projections $r_j \in \mathcal{A}$, or more precisely, to a projection $\sum_1^n r_j \otimes e_{jj}$ with $r_j \in \mathcal{A}$ for every j . Thus $\hat{p} = \sum_i^n \hat{r}_j$ and for each j , $\hat{r}_j = x_{j'}$ for some j' . As a consequence, $\text{Aff } T(\mathcal{A})$ is separable. Recall that $\text{Aff } T(\mathcal{A})$ is a real Banach space.

Now we follow the standard proof that the cardinality of any Hamel basis of an infinite dimensional separable Banach space is not countable. Choose a unit length $y_1 \in \{x_j\}$ such that $y_1 \notin \text{span}\{x_1\}$, and then choose $y_2 \in \{x_j\} \setminus \text{span}\{x_1, x_2, y_1\}$. Recursively, construct a sequence $y_k \in \{x_j\}$ such that $\|y_k\| = 1$ and

$$y_k \notin M_k := \text{span}\{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{k-1}\}.$$

Since $\text{Ext}(T(\mathcal{A}))$ is infinite by hypothesis, $\text{Aff } T(\mathcal{A})$ has infinite dimension too and hence also $\text{span}\{x_j\}$ has infinite dimension. Thus the construction cannot terminate after a finite number of steps and hence the sequence $\{y_k\}$ is infinite.

As M_k is a closed subspace, $\delta_k := \text{dist}(y_k, M_k) > 0$ for all k . Choose an increasing sequence of integers n_k such that $2^{n_{k+1}} > \frac{2^{n_k+1}}{\delta_k}$ and let

$$y := \sum_1^\infty \frac{y_j}{2^{n_j}}.$$

Then $y \in \text{Aff } T(\mathcal{A})$. Now $y_j = x_{\pi(j)} = \hat{p}_{\pi(j)}$ for some index $\pi(j)$. Set $a := \sum_1^\infty \frac{p_{\pi(j)}}{2^{n_j}}$. Then $a \in \mathcal{A}_+$ and for every $\tau \in T(\mathcal{A})$ we have

$$\tau(a) = \sum_1^\infty \tau\left(\frac{p_{\pi(j)}}{2^{n_j}}\right) = \sum_1^\infty \frac{\hat{p}_{\pi(j)}(\tau)}{2^{n_j}} = \sum_1^\infty \frac{y_j(\tau)}{2^{n_j}} = y(\tau).$$

Reasoning by contradiction, assume that a is a linear combination of projections $r_i \in \mathcal{A}$, say $a = \sum_{i=1}^m \lambda_i r_i$. Since $a = \sum_{i=1}^m \frac{\lambda_i + \bar{\lambda}_i}{2} r_i$, assume that $\lambda_j \in \mathbb{R}$. Then for every $\tau \in T(\mathcal{A})$ we would also have

$$y(\tau) = \tau(a) = \sum_{i=1}^m \lambda_i \tau(r_i).$$

For each i , $r_i \in [p_{i'}]$ for some i' and hence $\tau(r_i) = \hat{p}_{i'}(\tau) = x_{i'}(\tau)$. Thus

$$y = \sum_{i=1}^m \lambda_i x_{i'} \in \text{span}\{x_j\}.$$

But then $y \in M_k$ for some k . Since $\sum_{j=1}^{k-1} \frac{y_j}{2^{n_j}} \in M_k$ by the definition of M_k , it follows that

$$\begin{aligned}
\frac{\delta_k}{2^{n_k}} &= \text{dist}\left(\frac{y_k}{2^{n_k}}, M_k\right) \\
&= \text{dist}\left(y - \sum_{j=1}^{k-1} \frac{y_j}{2^{n_j}} - \sum_{j=k+1}^{\infty} \frac{y_j}{2^{n_j}}, M_k\right) \\
&= \text{dist}\left(-\sum_{j=k+1}^{\infty} \frac{y_j}{2^{n_j}}, M_k\right) \\
&\leq \left\| \sum_{j=k+1}^{\infty} \frac{y_j}{2^{n_j}} \right\| \\
&\leq \sum_{j=k+1}^{\infty} \frac{1}{2^{n_j}} \\
&\leq \frac{2}{2^{n_{k+1}}}
\end{aligned}$$

a contradiction. \square

Recall from Remark 2.7 that if \mathcal{A} is simple, unital, non-elementary, of real rank zero, stable rank one, with strict comparison of projections (which implies both that $K_0(\mathcal{A})$ is weakly unperforated and that $T(\mathcal{A}) \neq \emptyset$), then $D(\mathcal{A} \otimes \mathcal{K})$ is dense in $\text{Aff } T(\mathcal{A})_+$. Examples of C^* -algebras satisfying the conditions in Proposition 5.1 can be found in the category of simple unital AF-algebras. Indeed, by a result of Blackadar [3], every Choquet simplex can be realized as the tracial state space of some simple unital AF-algebra. Thus it is enough to start with a Choquet simplex with infinitely many extremal points, e.g., the Brauer simplex with extreme boundary $[0, 1]$. Explicit examples can be found among crossed products coming from Cantor minimal systems [20].

Remark 5.2. *Real rank zero C^* -algebras satisfying the conditions of Proposition 5.1 provide a negative answer to Marcoux's questions [30, Question 1 and 2] on whether the span of the projections in a simple unital C^* -algebra must be closed and on whether if the span is dense, it must coincide with the algebra.*

5.2. Not LP but with “many projections”. C^* -algebras can fail to be the span of their projections even if they contain a “many projections”.

A C^* -algebra has the LP property if the span of its projections is dense. Of course, real rank zero algebras have the LP property and projectionless algebras do not. But it may be interesting to note that there are algebras with “many projections”, e.g., having the same ordered K_0 group as a real rank zero algebra and satisfying the SP property (every hereditary subalgebra contains a nonzero projection) and yet fail to satisfy the LP property.

Remark 5.3. *Let \mathcal{A} be a C^* -algebra with two distinct tracial states τ, τ' such that $\tau(p) = \tau'(p)$ for every projection $p \in \mathcal{A}$. Then \mathcal{A} does not have the LP property.*

Proof. Since τ and τ' agree on the projections of \mathcal{A} and hence on their linear combinations, they must agree also on the norm closure of span of the projections of \mathcal{A} . Hence the latter cannot coincide with \mathcal{A} . \square

An example of a “nice” algebra having two distinct tracial states that agree on all the projections of the algebra is an AI -algebra \mathcal{A} , (an inductive limit of finite direct sums of the form $\mathbb{M}_{n_1}(C[0, 1]) \oplus \mathbb{M}_{n_2}(C[0, 1]) \oplus \dots \oplus \mathbb{M}_{n_k}(C[0, 1])$) constructed by [37] (a special case of [38]) which is simple, unital, with $K_0(\mathcal{A}) = \mathbb{Q}$ (rational numbers), $K_0(\mathcal{A})_+ = \mathbb{Q}_+$ (positive rationals), $[I_{\mathcal{A}}] = 1 \in \mathbb{Q}$ and has tracial simplex $T(\mathcal{A}) \cong [0, 1]$ having two extreme points τ, τ' . It follows that $\tau(p) = \tau'(p)$ for every projection $p \in \mathcal{A}$. It can also be shown that \mathcal{A} has the SP property.

5.3. Non-unital algebras. If we relax the condition that \mathcal{A} is unital, we also see that \mathcal{A} may fail to be the span of its projections. The simplest example is provided by the algebra \mathcal{K} where infinite rank operator clearly cannot be a linear combination of projections in \mathcal{K} , which are finite.

More generally, no simple stable σ -unital C^* -algebra of real rank zero with non-empty tracial simplex $T(\mathcal{A})$ can be the span of its projections. To see this, first recall that F. Combes showed in an early work [12, Proposition 4.1 and Proposition 4.4] that every semifinite (also called densely defined) lower semicontinuous weight τ on a C^* -algebra \mathcal{A} has an extension to a normal weight $\bar{\tau}$ on the enveloping von Neumann algebra \mathcal{A}^{**} and if the weight is tracial, then the extension is unique. More recently, Ortega, Rordam, and Thiel proved in [32, Proposition 5.2] that if the weight τ is tracial then the extension $\bar{\tau}$ is also tracial. Notice that the faithfulness of τ does not guarantee the faithfulness of $\bar{\tau}$. However, if τ is faithful, $Q \in \mathcal{A}^{**}$ is an open projection, and $\bar{\tau}(Q) = 0$, then $Q = 0$.

Notice also that for every element $a \in \mathcal{A}$, the range projection R_a of a as an element in the enveloping von Neumann algebra \mathcal{A}^{**} is an open projection. Indeed

$$R_a = \chi_{aa^*}(0, \|a\|^2] = \chi_{aa^*}(0, \|a\|^2 + 1)$$

where χ_{aa^*} denotes the spectral measure of aa^* in \mathcal{A}^{**} . R_a may fail to belong to $\mathcal{M}(A)$ (actually, most likely it is not).

Lemma 5.4. *Let \mathcal{A} be a σ -unital simple C^* -algebra of real rank zero. No element $a \in \mathcal{A}$ with $\bar{\tau}(R_a) = \infty$ for at least one $\tau \in T(\mathcal{A})$ can be a linear combination of projections. Such an element a always exists in $\mathcal{A} \otimes \mathcal{K}$ when $T(\mathcal{A}) \neq \emptyset$.*

Proof. Assume that $a = \sum_{i=1}^m \lambda_i p_i$ for some scalars λ_i and projections $p_i \in \mathcal{A}$. Then $R_a \leq \bigvee_{i=1}^m p_i$ where the supremum is of course taken in \mathcal{A}^{**} . Then

$$\bar{\tau}(R_a) \leq \bar{\tau}\left(\bigvee_{i=1}^m p_i\right) \leq \sum_{i=1}^m \bar{\tau}(p_i) = \sum_{i=1}^m \tau(p_i) < \infty$$

where the second inequality is a well-known von Neumann algebra property derived from the Kaplanski parallelogram law.

To see the last statement, first one sees that a sub C^* -algebras of $\mathcal{A} \otimes \mathcal{K}$ is $*$ -isomorphic to \mathcal{K} by Brown’s Stabilization Theorem ([8]). Then $a = \sum_{i=1}^{\infty} \frac{1}{n} e_{ii}$ satisfies $\bar{\tau}(R_a) = \infty$ for all $\tau \in T(\mathcal{A})$. \square

5.4. Non-simple algebras. Again, the simplest example of “nice” non-simple algebras that fail to be the linear span of their projections is given by \mathcal{K} , or more precisely by the unitization $\mathcal{A} := \mathbb{C}I + \mathcal{K}$ of \mathcal{K} . This has been observed by Marcoux in [30]. Indeed, the collection of linear combinations of projections in \mathcal{A} is $\mathbb{C}I + \mathcal{F}$ where \mathcal{F} denotes the finite rank class and hence is a proper subset of \mathcal{A} .

6. POSITIVE COMBINATIONS OF PROJECTIONS

For each positive element a in a C^* -algebra \mathcal{A} of real rank zero, the hereditary C^* -subalgebra $\text{her}(a) := (a\mathcal{A}a)^-$ is also of real rank zero by [10], and hence, has a sequential approximate identity of projections $\{p_i\}$. In the enveloping von Neumann algebra \mathcal{A}^{**} , the sequence $\{p_i\}$ converges strongly to the range projection R_a of a . Furthermore,

$$\text{her}(a) = \text{her}(R_a) := (R_a\mathcal{A}^{**}R_a)^- \cap \mathcal{A}.$$

Given a nonzero open projection $Q \in \mathcal{A}^{**}$ with the property that $\bar{\tau}(Q) < \infty$ for every $\tau \in T(\mathcal{A})$, the evaluation map $T(\mathcal{A}) \ni \tau \rightarrow \hat{Q}(\tau) := \bar{\tau}(Q)$ is clearly affine. In the case that $\text{Ext}(T(\mathcal{A}))$ is finite, every affine map is also continuous and hence $\hat{Q} \in \text{Aff } T(\mathcal{A})$. Since Q is a nonzero open projection and every trace $\tau \in T(\mathcal{A})$ is faithful, it follows that $\inf_{\tau \in T(\mathcal{A})} \hat{Q} > 0$. To summarize, when $\text{Ext}(T(\mathcal{A}))$ is finite and $Q \in \mathcal{A}^{**}$ is an open projection, then

$$(6.1) \quad \bar{\tau}(Q) < \infty \quad \forall \tau \in T(\mathcal{A}) \quad \Rightarrow \quad \begin{cases} \sup_{\tau \in T(\mathcal{A})} \hat{Q}(\tau) < \infty \\ \inf_{\tau \in T(\mathcal{A})} \hat{Q}(\tau) > 0 \end{cases}$$

Theorem 6.1. *Let \mathcal{A} be a σ -unital simple separable C^* -algebra with real rank zero, stable rank one, strict comparison of projections and assume that $T(\mathcal{A})$ has a finite number of extremal points. Then an element $a \in \mathcal{A}_+$ is a positive combination of projections if and only if $\bar{\tau}(R_a) < \infty$ for all $\tau \in T(\mathcal{A})$.*

We will present the proof through the chain of the following lemmas. Our first result extends to the present setting the main tool that we used in [25] and [26].

Lemma 6.2. *Let \mathcal{A} be a simple C^* -algebra \mathcal{A} of real rank zero and stable rank one having strict comparison of projections and such that $T(\mathcal{A})$ has a finite set of extremal tracial states. Let p, q be projections in \mathcal{A} with $qp = 0$, $q \precsim p$ and let $b = qb = bq$ be a positive element of \mathcal{A} . Then for every scalar $\alpha > \|b\|$, the positive element $a := \alpha p \oplus b$ is a positive combination of projections.*

Proof. Let $r := p + q$, then the corner $r\mathcal{A}r$ of \mathcal{A} satisfies the same hypotheses as \mathcal{A} and it is unital. Thus by Theorem 4.4 there is a universal constant V_0 such that for every $a \in r\mathcal{A}r$, there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ and projections $p_1, p_2, \dots, p_n \in r\mathcal{A}r$ such that

$$a = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$$

and

$$|\alpha_1| + |\alpha_2| + \dots + |\alpha_n| \leq V_0 \|a\|.$$

This is precisely condition (1) of [25, Proposition 2.7]. Condition (2) of the same proposition, namely that the positive combinations of projections of $r\mathcal{A}r$ are norm dense in $(r\mathcal{A}r)_+$, is an immediate consequence of the hypothesis that $RR(\mathcal{A}) = 0$ and hence $RR(r\mathcal{A}r) = 0$. Thus the conclusion of [25, Proposition 2.7] applies, namely every positive invertible operator in $r\mathcal{A}r$ is a positive combination of projections. This permits to apply [25, Lemma 2.9] which yields the requested positive combination of projections. \square

Lemma 6.3. *Let \mathcal{A} be a stable σ -unital C^* -algebra of real rank zero, $\emptyset \neq T(\mathcal{A})$ has the strict comparison of projections, and let $Q \in \mathcal{A}^{**}$ be an open projection.*

- (i) $\sup_{\tau \in T(\mathcal{A})} \bar{\tau}(Q) < \infty$ if and only if there exists a projection $r \in \mathcal{A}$ such that $\bar{\tau}(Q) < \tau(r)$ for all $\tau \in T(\mathcal{A})$.
- (ii) Condition (i) is satisfied if and only if there is a projection $r \in \mathcal{A}$, an open projection $R' \leq r$, and a partial isometry $w \in \mathcal{A}^{**}$ with $ww^* = Q$ and $w^*w = R'$ such that the map $\phi(x) = w^*xw$ is a trace-preserving $*$ -isomorphism between $\text{her}(Q)$ and $\text{her}(R')$.
- (iii) If $\text{Ext}(T(\mathcal{A}))$ is finite and condition (ii) is satisfied, then the projection r can be chosen so that $\bar{\tau}(Q) < \tau(r) < 2\bar{\tau}(Q)$ for all $\tau \in T(\mathcal{A})$.

Proof.

(i) The sufficiency is obvious since $\bar{\tau}(Q) < \tau(r) = \hat{r}(\tau)$ implies that $\bar{\tau}(Q)$ is bounded since \hat{r} is a continuous function on the compact set $T(\mathcal{A})$.

Now we prove the necessity. The stability of \mathcal{A} guarantees that for every $\tau \in T(\mathcal{A})$ the trace $\bar{\tau}$ is infinite, that is $\bar{\tau}(I) = \infty$. Let $\{p_i\}$ be an approximate identity of \mathcal{A} consisting of projections. Then $p_j \uparrow I$ and hence $\tau(p_j) \uparrow \infty$. Thus $\{\hat{p}_i\}$ is a monotone increasing sequence of continuous functions on $T(\mathcal{A})$ with $\lim \hat{p}_i(\tau) = \infty$ pointwise. Since $T(\mathcal{A})$ is compact, it follows from the well-known Dini's Theorem in elementary topology that there exists an integer n_0 such that

$$\inf_{\tau \in T(\mathcal{A})} \tau(p_{n_0}) > \sup_{\tau \in T(\mathcal{A})} \bar{\tau}(Q).$$

Thus it is enough to set $r = p_{n_0}$.

- (ii) If $Q \sim R' \leq r$, then $\bar{\tau}(Q) = \bar{\tau}(R') \leq \tau(r)$ for all $\tau \in T(\mathcal{A})$. In order to obtain strict inequality, it is enough to replace r with a projection $r' \in \mathcal{A}$, $r' \not\geq r$. This proves the sufficiency.

For the necessity, assume that $\sup_{\tau \in T(\mathcal{A})} \bar{\tau}(Q) < \infty$ and let $r \in \mathcal{A}$ be the projection with $\bar{\tau}(Q) < \tau(r)$ for all $\tau \in T(\mathcal{A})$ provided by (i). Since the hereditary algebra $\text{her}(Q) = (Q\mathcal{A}^{**}Q) \cap \mathcal{A}$ has real rank zero [10] and is σ -unital, one can find an increasing approximate identity of projections $q_i \in \mathcal{A}$ for $\text{her}(Q)$ and setting $r_i = q_{i+1} - q_i$ obtain $Q = \bigoplus_{i=1}^{\infty} r_i$ and hence

$$\sum_{i=1}^{\infty} \tau(r_i) = \bar{\tau}(Q) \leq \sup_{\tau \in T(\mathcal{A})} \bar{\tau}(Q).$$

Since $\tau(r_1) < \tau(r)$ for all $\tau \in T(\mathcal{A})$, by the strict comparison of projections one can find a partial isometry $v_1 \in \mathcal{A}$ such that $v_1 v_1^* = r_1$ and $v_1^* v_1 = r'_1 < r$. Similarly, because of $\tau(r_2) < \tau(r - r'_1)$ for all $\tau \in T(\mathcal{A})$, one can find another partial isometry $v_2 \in \mathcal{A}$ such that $v_2 v_2^* = r_2$ and $v_2^* v_2 = r'_2 < r - r'_1$. Repeating the construction recursively, one obtains a sequence of partial isometries $\{v_i\} \subset \mathcal{A}$ with mutually orthogonal initial projections $\{r_i\}$ in $\text{her}(Q)$ and mutually orthogonal range projections $\{r'_i\}$ in $r\mathcal{A}r$. Define $R' := \sum_{i=1}^{\infty} r'_i$ and $w = \sum_{i=1}^{\infty} v_i$. Then $R' \in \mathcal{A}^{**}$ is an open projection, $R' \leq r$, $w \in \mathcal{A}^{**}$ is a partial isometry, and $ww^* = Q$ and $w^*w = R'$. Thus $Q \sim R'$. Define $\phi : \text{her}(Q) \rightarrow \mathcal{A}^{**}$ by $\phi(x) = w^*xw$. Using the fact that $q_j = \sum_{i=1}^j r_i$ converges in the strict topology to Q , it is now routine to show that $\phi(x) \in \mathcal{A}$ for every $x \in \text{her}(Q)$ and hence that ϕ is a trace-preserving $*$ -isomorphism from $\text{her}(Q)$ onto $\text{her}(R')$.

- (iii) By the hypothesis that $\text{Ext}(T(\mathcal{A}))$ is finite, it follows that the evaluation map $\hat{Q}(\tau) = \bar{\tau}(Q)$ is continuous on $T(\mathcal{A})$. Identify \mathcal{A} with $\mathcal{A} \otimes \mathcal{K}$ and choose $\{q_j\}$ to be an approximate identity of $\text{her}(Q \oplus Q)$ consisting of projections of $\mathcal{A} \otimes \mathcal{K}$. Then $\{\tau(q_j)\}$ increases to $\bar{\tau}(Q \oplus Q) = 2\bar{\tau}(Q)$, that is the sequence of continuous

functions \hat{q}_j increase pointwise to the continuous function $2\hat{Q}$. By Dini's theorem, the convergence is uniform. Thus choose $r_0 = q_{j_0}$ for an appropriate j_0 , as wanted. \square

The next lemma is the technical crux of the proof.

Lemma 6.4. *Let \mathcal{A} be a unital simple C^* -algebra \mathcal{A} of real rank zero and stable rank one with strict comparison of projections, and with finitely many extremal tracial states. Let $a \in \mathcal{A}_+$ be such that $\bar{\tau}(R_a) > \frac{1}{2}$ for all $\tau \in \mathbf{T}(\mathcal{A})$. Then a is a positive combination of projections.*

Proof. Assume without loss of generality that $\|a\| = 1$. Let $\{\tau_i\}_1^m$ be the collection of the extremal tracial states of \mathcal{A} . Notice that

$$(6.2) \quad I = \chi_{[0,1]}(a) = \chi_{\{0\}}(a) + R_a.$$

By the w^* -continuity of each $\bar{\tau}_i$,

$$\lim_{\lambda \rightarrow 0+} \bar{\tau}(\chi_{(0,\lambda)}(a)) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0+} \bar{\tau}(\chi_{(\lambda,1]}(a)) = \bar{\tau}(R_a).$$

Since by (6.2)

$$\bar{\tau}_i(\chi_{\{0\}}(a)) = 1 - \bar{\tau}_i(R_a) < \bar{\tau}_i(R_a) \quad \forall 1 \leq i \leq m,$$

we can find $0 < \alpha < \beta < 1$ such that

$$\bar{\tau}_i(\chi_{[0,\alpha)}(a)) < \bar{\tau}_i(\chi_{(\beta,1]}(a)) \quad \forall 1 \leq i \leq m$$

and hence

$$(6.3) \quad \bar{\tau}(\chi_{[0,\alpha)}(a)) < \bar{\tau}(\chi_{(\beta,1]}(a)) \quad \forall \tau \in \mathbf{T}(\mathcal{A}).$$

Choose arbitrary numbers $0 < \gamma_1 < \gamma_2 < \gamma_3 < \alpha < \gamma_4 < \beta$. Let $f : [0, 1] \rightarrow [0, 1]$ be the continuous function defined by

$$f(t) = \begin{cases} t & t \in [0, 1] \setminus [\gamma_1, \gamma_3] \\ \gamma_1 & t \in [\gamma_1, \gamma_2] \\ \text{linear} & t \in [\gamma_2, \gamma_3]. \end{cases}$$

Notice that $f(a) \geq 0$ and $R_{f(a)} = R_a$.

Recall that for every $0 \leq \gamma < \delta < \infty$, the projection $\chi_{[\gamma,\delta]}(a)$ is closed. Now by using the hypothesis that \mathcal{A} is unital, it follows that $\chi_{[\gamma,\delta]}(a)$ is compact. Notice also that $\chi_{(\gamma,\delta)}(a)$ is open and so is $\chi_{(\gamma,1]}(a) = \chi_{(\gamma,2)}(a)$. Thus by Brown's interpolation property [9], there exist projections $s, p, q \in \mathcal{A}$ such that

$$\begin{aligned} \chi_{[0,\gamma_1]}(a) &\leq s \leq \chi_{[0,\gamma_2)}(a) \\ \chi_{[0,\gamma_3]}(a) &\leq q \leq \chi_{[0,\alpha)}(a) \\ \chi_{[\beta,1]}(a) &\leq p \leq \chi_{(\gamma_4,1]}(a). \end{aligned}$$

It is immediate to see that

$$(6.4) \quad pq = 0, \quad s \leq q, \quad \text{and} \quad p \leq I - s.$$

Define the following elements of \mathcal{A} :

$$\begin{aligned} b &:= a - f(a) + sf(a)s \\ a_1 &:= b + \alpha p \\ a_2 &:= (I - s)f(a)(I - s) - \alpha p. \end{aligned}$$

Since the spectral projections of a commute with a and hence with $f(a)$, and since $s - \chi_{[0, \gamma_1]}(a) \leq \chi_{(\gamma_1, \gamma_2)}(a)$, it follows that $s - \chi_{[0, \gamma_1]}(a)$ commutes with both $f(a)(\chi_{[0, \gamma_1]}(a) + \chi_{[\gamma_2, 1]}(a))$ and with $\gamma_1 \chi_{(\gamma_1, \gamma_2)}(a)$. Hence it commutes with

$$f(a) = f(a)(\chi_{[0, \gamma_1]}(a) + \chi_{[\gamma_2, 1]}(a)) + \gamma_1 \chi_{(\gamma_1, \gamma_2)}(a).$$

But then it follows that also s and hence $I - s$ commute with $f(a)$. As a consequence we have that

$$b = a - f(a) + f(a)s \quad \text{and} \quad a_2 = f(a) - f(a)s - \alpha p$$

and hence

$$a = a_1 + a_2.$$

Now $b \geq 0$ because $a \geq f(a) \geq 0$. Since

$$a - f(a) = (a - f(a))\chi_{[\gamma_1, \gamma_3]}(a) \leq \chi_{[\gamma_1, \gamma_3]}(a) \leq q$$

and

$$f(a)s \leq s \leq q,$$

hence $b = qbq$. Moreover

$$\begin{aligned} \|a - f(a)\| &= \gamma_2 - \gamma_1 \\ \|f(a)s\| &\leq \|f(a)\chi_{[0, \gamma_2]}(a)\| \leq \gamma_1, \end{aligned}$$

and hence $\|b\| \leq \gamma_2 < \alpha$. Now by (6.3) we have

$$\tau(q) = \bar{\tau}(q) \leq \bar{\tau}(\chi_{[0, \alpha]}(a)) < \bar{\tau}(\chi_{[\beta, 1]}(a)) \leq \tau(p) \quad \forall \tau \in \mathbf{T}(\mathcal{A}).$$

By the strict comparison of projections we obtain that $q \precsim p$.

As $a_1 = b + \alpha p$ with $b = qbq \geq 0$, $\|b\| < \alpha$, $qp = 0$ and $q \precsim p$, we obtain by Lemma 6.2 that a_1 is a positive combination of projections in \mathcal{A} .

We prove now that the same holds for a_2 . Notice first that by (6.4)

$$(6.5) \quad R_{a_2} \leq I - s \in \mathcal{A}.$$

Since

$$\chi_{[0, \gamma_2]}(a) - s \leq \chi_{[0, \gamma_2]}(a) - \chi_{[0, \gamma_1]}(a) = \chi_{(\gamma_1, \gamma_2)}(a)$$

and since $f(a)\chi_{(\gamma_1, \gamma_2)}(a) = \gamma_1 \chi_{(\gamma_1, \gamma_2)}(a)$, it follows that

$$f(a)(\chi_{[0, \gamma_2]}(a) - s) = \gamma_1(\chi_{[0, \gamma_2]}(a) - s).$$

Thus

$$\begin{aligned} a_2 &= f(a)(I - s) - \alpha p \\ &\geq f(a)(I - s) - \alpha \chi_{(\gamma_4, 1]}(a) \\ &= f(a)(\chi_{[0, \gamma_2]}(a) - s) + f(a)\chi_{(\gamma_2, \gamma_4]}(a) + (f(a) - \alpha)\chi_{(\gamma_4, 1]}(a) \\ &= \gamma_1(\chi_{[0, \gamma_2]}(a) - s) + f(a)\chi_{(\gamma_2, \gamma_4]}(a) + (f(a) - \alpha)\chi_{(\gamma_4, 1]}(a) \\ &\geq \gamma_1(\chi_{[0, \gamma_2]}(a) - s) + \gamma_2 \chi_{(\gamma_2, \gamma_4]}(a) + (\gamma_4 - \alpha)\chi_{(\gamma_4, 1]}(a) \\ &\geq \min\{\gamma_1, \gamma_4 - \alpha\} \left(\chi_{[0, \gamma_2]}(a) - s + \chi_{(\gamma_2, \gamma_4]}(a) + \chi_{(\gamma_4, 1]}(a) \right) \\ &= \min\{\gamma_1, \gamma_4 - \alpha\} (I - s). \end{aligned}$$

Thus $R_{a_2} \geq I - s$ and by (6.4) it follows that $R_{a_2} = I - s$ and

$$a_2 \geq \min\{\gamma_1, \gamma_4 - \alpha\} R_{a_2},$$

i.e., a_2 is locally invertible. But then by [25, Lemma 2.9] (see also proof of Lemma 6.2) a_2 is a positive combination of projections in \mathcal{A} . This concludes the proof. \square

By using these lemmas we can now provide the proof of Theorem 6.1.

Proof of the theorem. By Lemma 5.4 we see that the condition that $\bar{\tau}(R_a) < \infty$ for all $\tau \in T(\mathcal{A})$ is necessary.

To prove the sufficiency, assume first that \mathcal{A} is stable. Then by Lemma 6.3 (iii) there is a projection $r \in \mathcal{A}$ such that $\bar{\tau}(R_a) < \tau(r) < 2\bar{\tau}(R_a)$ for all $\tau \in T(\mathcal{A})$ and by part (ii) of the same lemma, and in the notations of the lemma, there is a trace preserving $*$ -isomorphism ϕ from $\text{her}(a) = \text{her}(R_a)$ onto $\text{her}(R') = \text{her}(\phi(a))$ with $R' \leq r$. Decomposing $\phi(a)$ into a positive combination of projections (necessarily in $\text{her}(\phi(a))$) is equivalent to decomposing a into a positive combination of projections (necessarily in $\text{her}(a)$). Thus we can thus assume without loss of generality that $R_a \leq r$. By passing to the corner $r\mathcal{A}r$ of \mathcal{A} , which satisfies the same properties as \mathcal{A} , we can further assume that \mathcal{A} is unital, i.e., identify r with I . By renormalizing the trace τ , we thus have that $\bar{\tau}(R_a) > \frac{1}{2}$. But then the conclusion that a is a positive combination of projections follows from Lemma 6.4.

Finally, we remove the condition that \mathcal{A} is stable. Let $a \in \mathcal{A}_+$ satisfy the condition $\bar{\tau}(R_a) < \infty$ for all $\tau \in T(\mathcal{A})$. Since R_a is an open projection, $R_a = \sum_{j=1}^{\infty} r_j$ for some projections $r_j \in \mathcal{A}$ and the series converges in the strict topology. Then

$$\begin{aligned} (\overline{\tau \otimes \text{Tr}})(R_{a \otimes e_{11}}) &= (\overline{\tau \otimes \text{Tr}})(R_a \otimes e_{11}) = (\overline{\tau \otimes \text{Tr}})\left(\sum_{j=1}^{\infty} r_j \otimes e_{11}\right) \\ &= \sum_{j=1}^{\infty} (\overline{\tau \otimes \text{Tr}})(r_j \otimes e_{11}) = \sum_{j=1}^{\infty} (\tau \otimes \text{Tr})(r_j \otimes e_{11}) \\ &= \sum_{j=1}^{\infty} \tau(r_j) = \bar{\tau}\left(\sum_{j=1}^{\infty} r_j\right) = \bar{\tau}(R_a). \end{aligned}$$

Since $\mathcal{A} \otimes \mathcal{K}$ satisfies the same conditions as \mathcal{A} and is stable, it follows from the first part of the proof that $a \otimes e_{11}$ is a positive combination of projections belonging to $\text{her}(a \otimes e_{11}) = \text{her}(a) \otimes e_{11}$. But then a too is a positive combination of projections in $\text{her}(a)$. \square

If \mathcal{A} is unital, for every $a \in \mathcal{A}_+$ it follows that $\bar{\tau}(R_a) \leq \bar{\tau}(I) = 1$ for all $\tau \in T(\mathcal{A})$. Thus a is a positive linear combination of projections by Theorem 6.1.

Corollary 6.5. *Let \mathcal{A} be a unital simple separable C^* -algebra with real rank zero, stable rank one, strict comparison of projections and assume that $T(\mathcal{A})$ has a finite number of extremal points. Then every element $a \in \mathcal{A}_+$ is a positive linear combination of projections in \mathcal{A} .*

Corollary 6.6. *Let \mathcal{A} be a σ -unital simple separable C^* -algebra with real rank zero, stable rank one, strict comparison of projections and assume that $T(\mathcal{A})$ has a finite number of extremal points. Then an element $a \in \mathcal{A}$ is a linear combination of projections if and only if $\bar{\tau}(R_a) < \infty$ for all $\tau \in T(\mathcal{A})$.*

Proof. The necessity is given by Lemma 5.4. Assume that $\bar{\tau}(R_a) < \infty$ for all $\tau \in T(\mathcal{A})$ and let $Q := R_{aa^*+a^*a}$. Since $Q = R_a \vee R_{a^*}$ and since $R_a \sim R_{a^*}$ in \mathcal{A}^{**} , it follows that $\bar{\tau}(Q) \leq 2\bar{\tau}(R_a) < \infty$ for every $\tau \in T(\mathcal{A})$. Now a decomposes naturally into a linear combination of positive elements of \mathcal{A} , all with range projections

dominated by Q and hence all satisfying the condition of Theorem 6.1. Thus they are all positive combinations of projections and hence a is a linear combination of projections. \square

We would like to point out why for unital algebras we first proved that every element is a linear combination of projections and from that we deduced that every positive element is a positive combination of projections, while in the non-unital case we first proved that positive decompositions hold for positive elements. In other words, why employing directly Theorem 4.4 for the proof of Corollary 6.6 would not suffice. Indeed while even in the non-unital case there is still a trace-preserving $*$ -isomorphism ϕ from $\text{her}(aa^* + a^*a)$ onto a (hereditary) subalgebra of a corner $r\mathcal{A}r$ for some projection $r \in \mathcal{A}$, we would obtain from Theorem 4.4 only that $\phi(a)$ is a linear combinations of projections in $r\mathcal{A}r$ but then we could not guarantee that those projections belong to $\phi(\text{her}(aa^* + a^*a))$ and hence that they can be carried back to $\text{her}(aa^* + a^*a)$ and provide a decomposition for a itself.

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